Lecture 10

Public Key Cryptography: Encryption + Signatures

Speeding up RSA decryption

Let : C - RSA ciphertext

$d_p = d \bmod (p - 1)$

$d_q = d \bmod (q - 1)$

compute:

$M_p = C^{d_p} \bmod p$

$M_q = C^{d_q} \bmod q$

$M = [M_p q(q^{-1} \bmod p) + M_q p(p^{-1} \bmod q)] \bmod (pq)$

and solve:

$M = M_p \bmod p$

$M = M_q \bmod q$

More on RSA

- **Modulus n is unique per user → can't share n**
- **What happens if Alice and Bob share the same modulus?**
  - **Alice has (e',d',n) and Bob – (e'',d'',n)**
  - **Alice wants to compute d'' (Bob's private key)**
  - **She knows that: e' * d'= 1 mod phi(n)**
  - **So: e' * d' = k * phi(n) + 1 and: e' * d' - 1 = k * phi(n)**
  - **Alice just needs to compute inverse of e'' mod X**
    - **where X = e' * d' – 1 = k * phi(n)**
    - **let's call this inverse d'''**
    - **and remember that: d''' * e'' = k' * k * phi(n) + 1**
    - **can we be sure that: d''' = d'' ?**
  - **Is it possible that e'' has no inverse mod X?**
    - **Yes, if e''=phi(n) or gcd(e'',k)>1 but this is very, very UNLIKELY!**
  - **For all decryption purposes, d''' is EQUIVALENT to d''**
  - **Suppose Eve encrypted for Bob: C = (m)$^{e''}$ mod n**
  - **Alice computes:**

$$C^{d'''} \bmod n = m^{e''d'''} \bmod n = (m)^{k' * k * \phi(n) + 1} \bmod n = m$$

El Gamal PK cryptosystem (83)

p - large prime
b - base, primitive element, generator
x - private exponent
y - public residue; $y \equiv b^x \bmod p$
$P = Z_p^*$
$C = Z_p^* \times Z_p^*$
publics : p, b, y
secrets : x

Encryption :
1. *generate random* $r \in Z_{p-1}$
2. *compute* : $k = b^r \bmod p$
3. *compute* : $c = my^r \bmod p = mb^{xr} \bmod p$
4. *ciphertext* $= \{k, c\}$

Decryption :
1. *compute* $k^x \bmod p$
2. *compute* $(k^x)^{-1} \bmod p$
3. $m' = (k^x)^{-1} c = b^{-rx} mb^{xr} \bmod p = m$

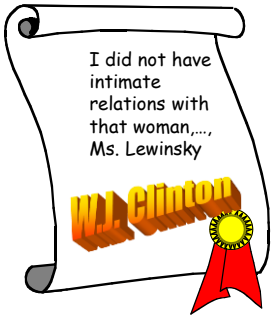
El Gamal (example)

$p = 13$
$b = 2$
$x = 9$
$y = 2^9 \bmod 13 = 5$

Encryption :
$m = 11$
$r = 10$
$k = 2^{10} \bmod 13 = 10$
$c = 11 * 5^{10} \bmod 13 = 2$
$\text{ciphertext} = \{10,2\}$

Decryption :
$10^9 \bmod 13 = 12$
$12^{-1} \bmod 13 = 12$
$2 * 12 = 24 \equiv 11 \bmod 13$

Digital Signatures



- Integrity
- Authentication
- Non-repudiation
- Time-stamping
- Causality
- Authorization

Digital Signatures

A signature scheme:

Usually message hash

(P,A,K,Sign,Verify)

***P* - plaintext (msgs)**

***A* - signatures**

***K* - keys**

Sign* - signing function: (*P*K*)->*A

Verify* - verification function: *(P*A*K)* → *{0,1}

7

RSA Signature Scheme

Use the fact that, in RSA, encryption reverses "decryption"

Let $n = pq$ where $p \neq q$ are two (large) primes

$e \in Z^*_{\Phi(n)}$ and $e = d^{-1} \bmod \Phi(n)$ and $ed \equiv 1 \bmod \Phi(n)$

$\Phi(n) = (p-1)(q-1)$

$Secrets: p, q, d$

$Publics: n, e$

Signing : $message = m$

$Sign(m): y = m^d \ mod \ n$

Verification : $signature = y$

$Verify(y, m): (m = y^e)$???

8

RSA Signature Scheme (contd)

- The good:
  - Verification can be cheap (like RSA encryption)
  - Mechanically same as RSA decryption function
  - Security based on RSA encryption
  - Signing is harder but #verify-s > 1…
  - Deterministic
- The bad:
  - Recall that RSA is malleable: signatures can be "massaged"
  - Phony "random" signatures
    - compute Y=RSA(e,X)=X^e mod n
    - X is a signature of Y because Y^d=X mod n
- The ugly:
  - Signing requires integrity!
  - How to sign multiple blocks?
  - Deterministic – needs additional randomization!

9

El Gamal Signature Scheme

p - *large prime*
b - *base, generator*
x - *private exponent*
y - *public residue*; $y \circ b^x \bmod p$
$P = Z_p^*$
$A = Z_p^* \times Z_p^*$
publics : p, b, y
secrets : x

Signing :
1. *generate random* $r \in Z_{p-1}$
2. *compute* : $k = b^r \bmod p$
3. *compute* : $c = (m - xk)r^{-1} \bmod p - 1$
4. *signature* = $\{k, c\}$

Verifying :
$y^k k^c \bmod p = b^m \bmod p$???

notice that :
$y^k k^c = b^{xb^r} (b^r)^{(m/r - xk/r)} = b^{xb^r + m - xb^r} = b^m$

10

El Gamal PK Cryptosystem

p - *large prime*
b - *base, primitive element, generator*
x - *private exponent*
y - *public residue*; $y = b^x \bmod p$
$P = Z_p^*$
$C = Z_p^* \times Z_p^*$
publics : p, b, y
secrets : x

Encryption :
1. *generate random* $r \in Z_{p-1}^*$
2. *compute* : $k = b^r \bmod p$
3. *compute* : $c = my^r \bmod p = mb^{xr} \bmod p$
4. *ciphertext* = $\{k, c\}$

Decryption :
1. *compute* $k^x \bmod p$
2. *compute* $(k^x)^{-1} \bmod p$
3. $m' = (k^x)^{-1} c = b^{-rx} m b^{xr} \bmod p = m$

El Gamal Signature Scheme

p - *large prime*
b - *base, generator*
x - *private exponent*
y - *public residue*; $y = b^x \bmod p$
$P = Z_p^*$
$A = Z_p^* \times Z_p^*$
publics : p, b, y
secrets : x

Signing :
1. *generate random* $r \in Z_{p-1}^*$
2. *compute* : $k = b^r \bmod p$
3. *compute* : $c = (m - xk) r^{-1} \bmod p - 1$
4. *signature* = $\{k, c\}$

Verifying :
$y^k k^c \bmod p = b^m \bmod p$???

notice that :
$y^k k^c = b^{xb^r} (b^r)^{(m/r - xk/r)} = b^{xb^r + m - xb^r} = b^m$

El Gamal Signature Scheme (contd)

The good:
- Signing is cheap(er)
- Designed as a signature function
- Non-deterministic (randomized)

The bad:
- Need GOOD source of random numbers
- Randomizers cannot be revealed (trace)
- Randomizers cannot be reused

The Digital Signature Standard (DSS)

- **Why DSS?**
- **RSA issues: patents, malleability, etc.**
- **A variant of El Gamal**
- **Originally for |p|=512 bits, now up to 1024**
- **Optimized for signature size (320- vs. 1024-bit)**
- **Signing - 1 exp, verification - 2 exps**
- **No attacks thus far**

13

DSS (contd)

p - large prime
b - base, generator
x - private exponent
y - public residue; $y = b^x \bmod p$
$P = Z_p^*, A = Z_p^* \times Z_p^*$
publics : p, b, y secrets : x

Signing :
1. generate random $r \in Z_{p-1}^*$
2. compute : $k = b^r \bmod p$
3. compute : $c = (m - xk) r^{-1} \bmod p - 1$
4. signature = $\{k, c\}$

Verifying :
$y^k k^c \bmod p = b^m \bmod p$???

p - 512 - bit prime
q - 160 - bit prime, $(p - 1) \% q = 0$
b - base, $b^q = 1 \bmod p$ $(b = d^{(p-1)/q})$
x - private exponent
y - public residue; $y = b^x \bmod p$
$P = Z_p^*, A = Z_q \times Z_q$
publics : p, q, b, y secrets : x

Signing :
1. generate random $r \in Z_{q-1}^*$
2. compute : $k = (b^r \bmod p) \bmod q$
3. compute : $c = (m + xk) r^{-1} \bmod q$
4. signature = $\{k, c\}$

Verifying :
$(b^{mc^{-1}} k^{kc^{-1}} \bmod p) \bmod q = b^k \bmod p$???

notice that :
$b^{mc^{-1}} y^{kc^{-1}} = b^{mr/(m+xb^r)} (b^x)^{(b^r r/(m+xb^r))}$
$= b^{(mr+xb^r r)/(m+xb^r)} = b^r$

14